On life-time-distributions of some one-dimensional diffusions and related exponential families

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Abstract

Diffusions X on the positive halfaxis with elastic killing boundary at zero and inaccessible boundary at ∞ are considered. The life-time ζ is finite and $h^{\theta}(x) = E_x \exp(\theta\zeta)$, $n \geq 0$, is an θ-excessive function. The $h^{(\theta)}$-transforms of X define a family of diffusions, such that the life-time distributions of them form an exponential family of distributions, the inverse local times at zero form an exponential family of processes with independent increments and the spectral measures of them are connected simply by translation.

1 Introduction

1.1. Let us given a filtered statistical space $(\Omega, \mathcal{F}, \mathcal{F}_t, (P_\theta))$ with $t \geq 0$; $\theta \in \Theta \subseteq R$, $(\mathcal{F}_t)$ right-continuous and $\mathcal{F} = \bigvee_t \mathcal{F}_t$.
We shall suppose $0 \in \Theta$.
Moreover, assume $X = (X_t, t \geq 0)$ is a real valued process on $(\Omega, \mathcal{F})$ with $\mathcal{F}_t = \sigma(X_s, s \leq t)$, $t \geq 0$. Assume that X has independent stationary increments under every P_θ and that it holds $P_\theta(X_0 = 0) \equiv 1$. Let $(\alpha_\theta, \sigma_\theta^2, \nu_\theta)$ be its Lévy-characteristics, occuring in the characteristic function

$$E_\theta \exp[i\lambda X_t] = \exp[t(i\alpha_\theta\lambda - \frac{\sigma_\theta^2}{2}\lambda^2 + \int\limits_R \left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\right)\nu_\theta(dy))]$$

with $t > 0$, $\lambda \in R$, $\theta \in \Theta$.

1.2. We say that $\mathcal{P} = (P_\theta, \theta \in \Theta)$, together with X, forms an *exponential family* of processes with independent stationary increments, if

(i) $P_\theta(X_t \in dx) = \exp[\theta \cdot x - \psi(\theta)d]P_0(X_t \in dx)$, $t > 0$, $x \in R$, $\theta \in \Theta$ for some function $\psi(\cdot)$. This property is equivalent with every of the following ones (see Küchler, Küchler (1981)):

(ii) $P_{\theta,t} \ll P_{0,t}$ for every $t > 0$ and
$$\frac{dP_{\theta,t}}{dP_{0,t}} = \exp[\Theta X_t - \psi(\theta)t],$$
where $P_{\theta,t}$ denotes the restriction of P_θ to $\mathcal{F}_t$,

(iii) For every $t > 0$ the "last observation" X_t is a sufficient statistic for θ with respect to $\mathcal{F}_t$

(iv) $\sigma_\theta^2 \equiv \sigma_0^2$, $d\nu_\theta(y) = \exp[\theta y]d\nu_0(y)$, $y \in R$,
$$\alpha_\theta = \alpha_0 + \sigma_0^2 \cdot \theta + \int\limits_R \left(e^{\theta y} - 1 - 1\frac{\theta y}{1+y^2}\right)\nu_0(dy), \quad \theta \in \Theta.$$

This equivalence does not hold for more general processes, e.g. Markov processes.
1.3. Example: If X forms a Brownian motion with diffusion constant equals to one and drift

$$b(x) = \sqrt{2\theta}\tanh(\sqrt{2\theta}x), \quad \theta > 0, x \in R$$

under P_θ, then (iii) holds, but (i) and (ii) do not (see Küchler (1982)).

1.4. Thus, every of the properties (i)-(iii) (or slight changed variants of them) could be used to define exponential families of processes in the general case.

This was done by several authors, see e.g. Küchler (1982), Küchler, Soerensen (1989), Ycart (1989).
In Küchler (1982) it was shown, that a family of conservative Markov processes under some conditions of regularity has the property (iii) if and only if for every $\theta \in \Theta$ there is a number $\psi(\theta)$ and a strictly positive function g_θ on the state space with

$$(i') \quad P_\theta(X_t \in dy | X_0 = x) = \exp(-\psi(\theta)t)\frac{g_\theta(y)}{g_\theta(x)} P_0(X_t \in dy | X_0 = x).$$

Obviously it holds

$$\int g_\theta(y) P_0(X_t \in dy | X_0 = x) = \exp(\psi(\theta)t) g_\theta(x),$$

i.e., g_θ is $\psi(\theta)$-regular for X under P_0.

1.5. Asmussen (1989) considered nonconservative Markov processes with finite life-time ζ and obtains a family of Markov processes with a kind of property (iii), where the life-time distributions form an exponential family.

In this paper we shall present families of nonconservative diffusions on $[0,\infty)$ which have a property similar to (ii), the life-time distributions of which form an exponential family, and the inverse local times at zero form an exponential family of processes with independent stationary increments. Moreover, their spectral measures are simply connected by translation. This shows that also behind the processes with independent stationary increments there are connections between (i)-(iii), more precisely between different notions of exponential families.
Analogue results can also be obtained for gap- or quasi-diffusions (see Küchler (1986) for definitions), but the formulations will be slightly more complicate.

2 Diffusions, local times and spectral measures

Here we shall summarize some properties of one-dimensional diffusions which are necessary in the sequel. For details the reader is refered to Ito, McKean (1974), Kac, Krein (1974), Küchler (1986), Küchler, Neumann (1991).

2.1. Suppose $X = (\Omega, \mathcal{F}, (\mathcal{F}_t), X_t, \zeta, P_x)$ is a (regular) diffusion on $[0, L)$ with speed measure $m(\cdot)$ and scale $s(\cdot)$ in the sense of Ito, McKean (1974). Assume $s(0) = m(0-) = 0$, $s(L-) = \infty$ and let 0 be a killing, elastic killing or reflecting boundary.

The infinitesimale generator A of X in $L_2(m)$ can be given by the restriction of the generalized second order differential operator $D_m D_s$ to

$$D_A := \{f \in L_2(m) | D_m D_s f \in L_2(m), a \cdot D_s^- f(0) = f(0)\},$$

where $a \in [0,\infty]$ is fixed. (The number a is connected with the killing rate of X at zero.)
Because of $s(L-) = \infty$ we have for $a < \infty$ that

$$P_x(\zeta < \infty, X_{\zeta-} = 0) \equiv 1.$$

2.2. Remark: Obviously, for every $c > 0$ the pair $(cm, c^{-1}s)$ together with the boundary condition

$$\frac{a}{c} D_{(c^{-1}s)} f(0) = f(0)$$

leads to the same process X. It should be noted here already, that certain quantities of X as the coefficient in the boundary condition even mentioned but also transition densities, resolvent kernels, local times and spectral measures are uniquely determined only up to the factor c.

2.3. For every complex λ and $a \in (0, \infty]$ let φ_a and ψ be the solution of

$$D_m D_s \Phi + \lambda \Phi = 0 \tag{2.1}$$

satisfying the boundary conditions

$$\begin{aligned} \varphi_a(0, \lambda = 1, \qquad & D_s^- \varphi_a(0, \lambda) = \frac{1}{a} \\ \psi(0, \lambda) = 0, \qquad & D_s^- \psi(0, \lambda) = 1. \end{aligned}$$

The functions $\varphi := \varphi_\infty$ and ψ are called the *fundamental solution* of (2.1). If $\lambda < 0$ then $\varphi_a(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ are positive, increasing, and it holds

$$\int_0^L \varphi^{-2}(u, \lambda) s(du) < \infty.$$

Define

$$\chi(x, \lambda) := \varphi(x, \lambda) \int_x^L \varphi^{-2}(u, \lambda) s(du), \quad x \in [0, L),\ \lambda \in K,$$

(K denotes the set of complex numbers).
Then $\chi(\cdot, \lambda)$ satisfies (2.1) and for fixed $\lambda < 0$ it is positive and decreasing. Moreover, we have for $\lambda < 0$

$$D_s^- \chi(0, \lambda) = -1, \ \ \chi(0, \lambda) = \lim_{x \uparrow L} \frac{\psi(x, \lambda)}{\psi(x, \lambda)} \quad \text{and,}$$

because of $S(L-) = \infty$, $D_s^- \chi(L-, \lambda) = 0$.
The function $G(\cdot)$ defined by

$$G(\lambda) := \chi(0, \lambda), \quad \lambda \in K_- := K \backslash [0, \infty)$$

is called the *characteristic function* of (m, s) (more precisely, of $m \circ s^{-1}$).
By assumption we have

$$G(0) := \lim_{\lambda \uparrow 0} G(\lambda) = s(L-) = \infty. \tag{2.2}$$

Moreover, it holds

$$\chi(x, \lambda) = G(\lambda)\varphi(x, \lambda) - \psi(x, \lambda), \quad x \in [0, L),\ \lambda \in K_-. \tag{2.2a}$$

In the sequel we allow a to be zero and put $\varphi_0 := \psi$.

2.4. The Laplace transform of the first hitting time of y

$$\sigma_y := \inf\{t > 0 | X_t = y\}$$

is given by

$$E_x \exp(\lambda \sigma_y) = \Phi(y, \lambda)/\Phi(x, \lambda), \quad \lambda < 0, x, y \in [0, L) \tag{2.3}$$

with

$$\Phi = \varphi_a \text{ if } x \le y \text{ and } \Phi = \chi \text{ if } x \ge y.$$

2.5. The *Wronskian* $W = W(\lambda)$ is defined by

$$W(\lambda) := \chi(x,\lambda) D_s^- \varphi_a(x,\lambda) - \varphi_a(x,\lambda) D_s^- \chi(x,\lambda), \ \lambda \in K, x \in [0,L).$$

It is independent of x, and we have

$$\left. \begin{array}{lll} W(\lambda) = \frac{1}{a} G(\lambda) + 1 & \text{if} & a \in (0,\infty], \\ W(\lambda) = G(\lambda) & \text{if} & a = 0. \end{array} \right\} . \tag{2.4}$$

For all functions f, g such that $D_m D_s f$ and $D_m D_s g$ make sense *Lagrange's identity* holds:

$$\int_0^x [g D_m D_s f - f D_m D_s g] dm = [f D_s g - g D_s f]_{-0}^{x+0}. \tag{2.5}$$

2.6 For every $a \in [0,\infty]$ there exists the *transition density* $p(t,x,y)$:

$$P_x(X_t \in dy) = p(t,x,y) m(dy), \quad t > 0, x, y \in [0,L). \tag{2.6}$$

It satisfies $p(t,x,\cdot) \in D_A$ and

$$\frac{\partial}{\partial t} p(t,x,y) = D_m D_s p(t,x,y), \quad t > 0; x, y \in [0,L). \tag{2.7}$$

For the *resolvent kernel*

$$r_\lambda^{(a)}(x,y) := \int_0^\infty e^{\lambda t} p(t,x,y) dt, \ \ \lambda \in K_- \ \ x, y \in [0,L)$$

it holds

$$r_\lambda^{(a)}(x,y) = \frac{\varphi_a(x \wedge y, \lambda) \chi(x \vee y, \lambda)}{W(\lambda)}, \ \lambda \in K_-, \ \ x, y \in [0,L). \tag{2.8}$$

For every $a \in [0,\infty]$ the *spectral measure* $r = r_a$ of $D_m D_s|_{D_A}$ is defined to be a measure on $[0,\infty)$ (on $(0,\infty)$ if $a < \infty$) with

$$r_\lambda^{(a)}(x,y) = \int_0^\infty \frac{\varphi_a(x,u)\varphi_a(y,u)}{u - \lambda} \tau_a(du), \quad \lambda < 0; x, y \in [0,L). \tag{2.9}$$

It exists and is uniquely determined. Moreover, it holds (see Küchler, Neumann (1991))

$$\left(\frac{1}{a} + \frac{1}{G(\lambda)} \right)^{-1} = r_\lambda^{(a)}(0,0) = \int_0^\infty \frac{\tau_a(d\mu)}{u - \lambda}, \ \lambda < 0, a \in (0,\infty] \tag{2.10}$$

and

$$\frac{1}{G(\lambda)} = \int_0^\infty \left(\frac{1}{u} - \frac{1}{u - \lambda} \right) \tau_0(d\mu). \tag{2.11}$$

We have

$$G(\lambda) = r_\lambda^{(\infty)}(0,0) = \int_0^\infty \frac{\tau_\infty(d\mu)}{u - \lambda}, \ \ \lambda < 0 \tag{2.12}$$

and using $G(0-)=\infty$ we get from (2.9)

$$\int_0^\infty \frac{\tau_a(u)}{u} = a, \quad a \in (0,\infty]. \tag{2.13}$$

2.7 Fix $a=\infty$ and denote by $l=l(t,x)$ the *local time* of X defined by

$$\int_0^t f(X_s)ds = \int_0^L f(x)l(t,x)m(dx), \quad f \text{ bounded, measurable.}$$

The function $l(\cdot,x)$ is nondecreasing and $l(t,x)$ can be supposed to be continuous in (t,x), $t>0, x\in[0,L)$.

For every $x\in[0,L)$ let $l^{-1}(\cdot,x)$ be the right-continuous inverse of $l(\cdot,x)$. Then $(l^{-1}(t,x), t\geq 0)$ is a stricly increasing process with independent stationary increments. It holds

$$E_0 e^{\lambda l^{-1}(t,0)} = \exp\left(-\frac{t}{r_\lambda^{(\infty)}(0,0)}\right) = \exp\left(-\frac{t}{G(\lambda)}\right), \; t>0, \lambda<0. \tag{2.14}$$

Thus, the Lévy-measure $\nu(du)$ of $l^{-1}(\cdot,0)$ can be calculated from (2.11) and (2.14):

$$\nu(du) = \int_0^\infty e^{ul}\tau_0(dl)du, \quad u>0. \tag{2.15}$$

In particular, we get

$$E_0 e^{\lambda l^{-1}(t,0)} = \exp[t\Big(\int_0^\infty (e^{\lambda u}-1)\nu(du)\Big)], \quad t>0, \lambda<0. \tag{2.16}$$

Note that $\nu(\varepsilon,\infty))$ gives the intensity of jumps of $l^{-1}(\cdot,0)$ with size greater than ε, i.e. the intensity of excursions of X from 0 of length greater than ε in local time scale. Thus

$$Q_\varepsilon(A) := \frac{\nu(A\cap(\varepsilon,\infty))}{\nu((\varepsilon,\infty))}, \quad A \in \mathcal{B},$$

is the probability that a jump of $l^{-1}(\cdot,0)$ has a size in A conditioned it is greater than ε.

2.8. Examples: Assume X is a diffusion on $[0,L)$ with infinitesimale generator

$$\frac{\sigma^2(x)}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

Then we can choose (assuming that all occuring integrals exists)

$$dm(x) = \frac{2}{\sigma^2(x)}\exp\left(2\int_0^x \frac{\mu(y)}{\sigma^2(y)}dy\right)dx \;\text{ and}$$

$$ds(x) = \exp\left(-2\int_0^x \frac{\mu(y)}{\sigma^2(y)}dy\right)dx.$$

a) Brownian motion on $[0, \infty)$ with elastic killing at zero and with drift $\mu \leq 0$:
Put $\sigma^2(x) \equiv 1$, $\mu(x) \equiv \mu \leq 0$ and fix $a \in (0, \infty]$. Then it holds

$$G(\lambda) = (\mu + (\mu^2 - 2\lambda)^{\frac{1}{2}})^{-1}, \quad \lambda \in K_-,$$

$$\tau_a(du) = \frac{(2u - \mu^2)^{\frac{1}{2}}}{\pi[(\frac{1}{a} + \mu)^2 + 2u - \mu^2]} 1\!\!1_{[\frac{\mu^2}{2}, \infty)}(u) du, \quad u \geq 0, \tag{2.17}$$

and, if $\mu < -\frac{1}{a}$, τ_a has an additional mass of amount $-(\mu + \frac{1}{a})$ at the point $-\frac{1}{a}(\mu + \frac{1}{2a})$. For $a = 0$ (killing boundary at zero) it holds

$$\tau_0(du) = \frac{(8u - \mu^2)^{\frac{1}{2}}}{8\pi} 1\!\!1_{[\frac{\mu^2}{8}, \infty)}(u) du.$$

Thus, it follows

$$\nu(du) = e^{\frac{\mu^2}{8} \cdot u} \cdot u^{-\frac{1}{2}} \cdot \frac{1}{4\sqrt{2\pi}} du, \quad u > 0.$$

b) Ornstein-Uhlenbeck-process on $[0, \infty)$ with elastic killing boundary at zero:
Assume $\sigma^2(x) \equiv 1$, $\mu(x) = -\rho x$, $(\rho > 0, x \geq 0)$.
Then we get

$$G(\lambda) = \frac{1}{2\sqrt{\rho}} \; \frac{\Gamma(-\frac{\lambda}{2\rho})}{\Gamma(\frac{1}{2} - \frac{\lambda}{2\rho})} \qquad \lambda \in K_-,$$

where Γ denotes the Gamma-function.
The spectral measure $\tau_a(du)$ $(a \in [0, \infty])$ is concentrated in the points u_k, $k \geq 0$ with

$$0 \leq u_0 < u_1 < \ldots < u_n < \ldots$$

and $\lim\limits_{n \to \infty} u_n = \infty$, which are poles of $(\frac{1}{a} + \frac{1}{G(\cdot)})^{-1}$. It holds

$$\tau_a(\{u_k\}) = \operatorname{Res} \left(\frac{1}{a} + \frac{1}{G(\lambda)} \right)^{-1} \Big|_{\lambda = u_k}.$$

In particular we get

$$a = 0 : u_k = (2k+1)\rho, \tau_0(\{u_k\}) = \binom{2(k+1)}{k} \frac{\rho\sqrt{\rho}}{\sqrt{\pi} 2^{2k}}$$

$$a = \infty : u_k = 2\rho k, \;\; \tau_\infty(\{u_k\}) = \frac{\sqrt{\rho}}{\sqrt{\pi}} 2^{-k} \binom{2k}{k}.$$

2.9. Now we shall summarize some properties of the life-time distribution. We know, that for $a < \infty$ the life-time ζ is finite and that $X_{\zeta-} = 0$.

2.10. Lemma: *Assume $a \in (0, \infty)$. Then the Laplace transform $h^{(\theta)}(x)$ of ζ is given by*

$$h^{(\theta)}(x) := E_x \exp(\theta\zeta) = \frac{\chi(x, \theta)}{\chi(0, \theta)} \cdot \frac{\frac{1}{a}}{\frac{1}{a} + \frac{1}{G(\theta)}}, \; \theta \leq 0, x \in [0, L). \tag{2.18}$$

Proof: We have

$$P_x(\zeta > t) = \int\limits_0^L p(t, x, y) m(dy),$$

and thus we obtain using (2.6):

$$\begin{aligned}
E_x e^{\lambda\zeta} &= \int\limits_0^\infty e^{\lambda t} P_x(\zeta \in dt) = -\int\limits_0^\infty e^{\lambda t} \frac{d}{dt} P_x(\zeta > t)dt = \\
&= -e^{\lambda t} P_x(\zeta > t)|_0^\infty + \lambda \int\limits_0^\infty e^{\lambda t} P_x(\zeta > t)dt \\
&= 1 + \lambda \int\limits_0^L r_\lambda^{(a)}(x,y)m,(dy), \quad \lambda < 0, x \in [0,L).
\end{aligned}$$

Therefore we get from (2.8), (2.5) and (2.4)

$$\begin{aligned}
W(\lambda)\cdot E_x e^{\lambda\zeta} &= \lambda\chi(x,\lambda)\int\limits_0^x \varphi_a(y,\lambda)m(dy) + \lambda\varphi_a(x,\lambda)\int\limits_x^L \chi(y,\lambda)m(dy) + W(\lambda) \\
&= -\chi(x,\lambda)\int\limits_0^x dD_s^-\varphi_a(y,\lambda) - \varphi_a(x,\lambda)\int\limits_x^L dD_s^-\chi(y,\lambda) + W(\lambda) \\
&= -\chi(x,\lambda)D_s^-\varphi_a(x,\lambda) + \chi(x,\lambda)\frac{1}{a} + \varphi_a(x,\lambda)D_s^-\chi(x,\lambda) + W(\lambda) \\
&= \frac{1}{a}\chi(x,\lambda) = \frac{\chi(x,\lambda)}{\chi(0,\lambda)}\ \frac{\frac{1}{a}}{\frac{1}{a}+\frac{1}{G(\lambda)}}\cdot W(\lambda). \qquad \square
\end{aligned}$$

2.11. Corollary: *(i) The life-time distribution under P_0 is a mixed exponential distribution with*

$$P_0(\zeta \in dt) = \frac{1}{a}\int\limits_0^\infty u e^{-ut}\cdot \frac{\tau_a(du)}{u} dt. \tag{2.19}$$

(ii) The life-time distribution under P_x is the convolution of the distributions of the first hitting time σ_0 under P_x and the life time ζ under P_0.

Proof: Use (2.10) for (i). (ii) is obvious from (2.18). $\square$

2.12 Example: Brownian motion on $[0,\infty)$ with elastic killing at zero and drift $\mu \le 0$: We have

$$\begin{aligned}
P_0(\zeta \in dt) &= \Bigg[\frac{1}{a\pi}\int\limits_{\frac{\mu^2}{2}}^\infty \frac{e^{-ut}\cdot(2u-\mu^2)^{\frac{1}{2}}}{(\frac{1}{a^2}+\frac{2\mu}{a}+2u)}\,du \\
&\quad + 1\!\mathrm{I}_{(-\infty,-\frac{1}{a})}(\mu)\cdot\frac{1}{a}\cdot|\mu+\frac{1}{a}|e^{\frac{1}{a}(\mu+\frac{1}{2a})t}\Bigg]dt, \\
P_x(\sigma_0 \in dt) &= e^{\mu x}\frac{x}{\sqrt{2x}}\, t^{-\frac{3}{2}}\exp\Big[-\frac{1}{2}\Big(\frac{x^2}{t}+\mu^2 t\Big)\Big]dt.
\end{aligned}$$

Thus, the life-time distribution under P_x is the convolution of these two distributions.

3 The $h^{(\theta)}$-transformed diffusion

3.1. Consider the function $h^{(\theta)}(\cdot)$ defined above by

$$h^{(\theta)}(x) := E_x e^{\theta\zeta},\ \theta \le 0, x \in [0,L),$$

and fix an $\theta < 0$.
Because of

$$\begin{aligned} e^{\theta t} \cdot E_x h^{(\theta)}(X_t) &= E_x\left(E_x\left(e^{\theta(\zeta\circ\Theta_t+t)}|\mathcal{F}_t\right) \cdot 1\!\!1_{\{\zeta>t\}}\right) \\ &= E_x\left(e^{\theta\zeta} \cdot 1\!\!1_{\{\zeta>t\}}\right) \le E_x e^{\theta\zeta} = h^{(\theta)}(x), \quad x \in [0,L), \end{aligned}$$

and $\lim_{t\downarrow 0} E_x h^{(\theta)}(X_t) = h^{(\theta)}(x)$, $x \in [0,L)$, this function $h^{(\theta)}(\cdot)$ is θ-excessive (- θ-excessive in the terminology of Blumenthal-Getoor (1968)). In particular

$$P^{(\theta)}(t,x,dy) := e^{\theta t} P(t,x,dy)\frac{h^{(\theta)}(y)}{h^{(\theta)}(x)}, \quad t > 0,\, x,y \in [0,L) \tag{3.1}$$

forms the transition function of a new Markov process $X^{(\theta)}$, the so-called $h^{(\theta)}$-transformed process of X. From (3.1) and (2.6) it follows

$$P^{(\theta)}(t,x,dy) = e^{\theta t}\frac{p(t,x,y)}{h^{(\theta)}(x)h^{(\theta)}(y)}\,[h^{(\theta)}(y)]^2 m(dy).$$

Thus $X^{(\theta)}$ is a diffusion on $[0,L)$ again. the choose of speed measure and scale is not unique as was noted in Remark 2.2 above. We put

$$dm^{(\theta)}(x) = \rho^{(\theta)}(x)dm(x), \quad ds^{(\theta)}(x) = \rho^{(\theta)^{-1}}(x)ds(x) \tag{3.2}$$

with

$$\rho^{(\theta)}(x) = c^2(\theta)h^{(\theta)^2}(x)$$

where $c^2(\cdot)$ is a constant with respect to x but is allowed to depend on θ. It will be specified later. $m^{(\theta)}$ ($s^{(\theta)}$) is the *speed measure* (the *scale*, respectively) we have chosen for $X^{(\theta)}$.
Obviously, it holds

$$s^{(\theta)}(L-) = c^{-2}\int_0^L h^{(\theta)^{-2}}(x)s(dx) = \infty$$

(note that $h^{(\theta)}$ is decreasing and nonnegative if $\theta < 0$).
The infinitesimal generator $A^{(\theta)}$ of $X^{(\theta)}$ in $L_2(m^{(\theta)})$ is given by

$$A^{(\theta)}f = \frac{1}{h^{(\theta)}}Af \cdot h^{(\theta)} + \theta f \tag{3.3}$$

with

$$f \in D_{A^{(\theta)}} \text{ if and only if } f \cdot h^{(\theta)} \in D_A.$$

Thus, every $f \in D_{A^{(\theta)}}$ satisfies the boundary condition

$$\begin{aligned} &D_{S^{(\theta)}} f(0) = \rho^{(\theta)}(0)D_s f(0) = \rho^{(\theta)}(0)D_s\left(\frac{fh^{(\theta)}}{h^{(\theta)}}\right)(0) = \\ &= \rho^{(\theta)}\,\frac{f^{(\theta)}D_s(fh^{(\theta)}) - h^{(\theta)}fD_s h^{(\theta)}}{[h^{(\theta)}]^2}(0) = \\ &c^2(\theta)\cdot\left(h^{(\theta)^2}\cdot f\cdot\frac{1}{a} - fh^{(\theta)}D_s h^{(\theta)}(0)\right) = \\ &\rho^{(\theta)}(0)\left(\frac{1}{a} + \frac{1}{G(\lambda)}\right)f(0). \end{aligned}$$

Now it is plausible to choose $c(\theta) = h^{(\theta)^{-1}}(0)$, $\theta \in (-\infty, 0]$.
With

$$a_\theta := \left[\frac{1}{a} + \frac{1}{G(\theta)}\right]^{-1} \tag{3.4}$$

we obtain the boundary condition

$$a_\theta D_{s^{(\theta)}} f(0) = f(0), \quad f \in D_{A^{(\theta)}}. \tag{3.5}$$

3.2. Remark: As we have seen, for every $\theta < 0$ the transition function $P^{(\theta)}(t, x, dy)$ and therefore the processes $X^{(\theta)}$ are well-defined. But in the choose of $(m^{(\theta)}, s^{(\theta)})$ there is some arbitrariness because of the factor $c(\theta)$. It turns out that the most formula below become more simple if we choose $c(\theta) = [h^{(\theta)}(0)]^{-1}$. In particular we get $\rho^{(\theta)}(x) = \left[\frac{h^{(\theta)}(x)}{h^{(\theta)}(0)}\right]^2 = \frac{\chi^2(x,\theta)}{G^2(\theta)}$.

3.3. Proposition: *The characteristic function $G^{(\theta)}(\lambda)$ of $(m^{(\theta)}, s^{(\theta)})$ satisfies*

$$\frac{1}{G^{(\theta)}(\lambda)} = \frac{1}{G(\lambda+\theta)} - \frac{1}{G(\theta)}, \quad \lambda \in K_-. \tag{3.6}$$

Proof: We start with the computation of the fundamental solution $\varphi^{(\theta)}$ and $\psi^{(\theta)}$ of $(m^{(\theta)}, s^{(\theta)})$.

3.4. Lemma: *It holds*

$$\psi^{(\theta)}(x, \lambda) = \frac{h^{(\theta)}(0)}{h^{(\theta)}(x)} \psi(x, \lambda+\theta) = \frac{\psi(x, \lambda+\theta)}{\chi(x,\theta)} \cdot G(\theta), \tag{3.7}$$

$$\varphi^{(\theta)}(x, \lambda) = \left[\frac{G(\theta)\varphi(x, \lambda+\theta) - \psi(x, \lambda+\theta)}{\chi(x,\theta)}\right]. \tag{3.8}$$

Proof: To prove (3.7) note that $\psi^{(\theta)}$ is the unique solution on

$$\Phi(x,\lambda) = s^{(\theta)}(x) - \lambda \int_0^x (s^{(\theta)}(x) - s^{(\theta)}(u))\Phi(u,\lambda) m^{(\theta)}(dm)$$

or, equivalently,

$$\left[\frac{h^{(\theta)}(x)}{h^{(\theta)}(0)}\right]^2 D_s \Phi(x,\lambda) = 1 - \lambda \int_0^x \Phi(u,\lambda) \left[\frac{h^{(\theta)}(u)}{h^{(\theta)}(0)}\right]^2 m(du)$$

with $\Phi(0, \lambda) = 0$.
Using Lagrange's identity (2.5) it is easy to see that the right-hand side of (3.7) satisfies this equation. The equation (3.8) is proved similary. $\square$
Applying (3.7) and (3.8) we obtain

$$\frac{1}{G_\theta(\lambda)} = \lim_{x \uparrow L} \frac{\varphi^{(\theta)}(x,\lambda)}{\psi^{(\theta)}(x,\lambda)} = \frac{1}{G(\lambda+\theta)} - \frac{1}{G(\theta)}. \qquad \square$$

3.5. Now we are ready to calculate the other quantities connected with $X^{(\theta)}$: the spectral measure $\tau^{(\theta)}_{a^{(\theta)}}$ and the Lévy-measure $\nu^{(\theta)}$.

3.6. Proposition: *It holds*

$$\int_0^\infty \frac{d\tau^{(\theta)}_{a^{(\theta)}}(u)}{u-\lambda} = \int_0^\infty \frac{d\tau_a(u)}{u-(\lambda+\theta)} = \int_0^\infty \frac{\tau_a(du+\theta)}{u-\lambda} \tag{3.9}$$

i.e. $\tau^{(\theta)}_{a^{(\theta)}}(\cdot)$ *charges* du *as* $\tau_a(\cdot)$ *charges* $du+\{\theta\}$.

Proof: By (2.10), (3.4) and (3.6) we have

$$\int_0^\infty \frac{d\tau^{(\theta)}_{(a^{(\theta)}}(u)}{u-\lambda} = \frac{1}{\frac{1}{a_\theta}+\frac{1}{G_\theta(\lambda)}} = \frac{1}{\frac{1}{a}+\frac{1}{G(\theta)}+\frac{1}{G(\lambda+\theta}-\frac{1}{G(\theta)}} = \int_0^\infty \frac{d\tau_a(u)}{u-\lambda-\theta}. \quad \square$$

Consequently, the Lévy-measure $\nu^\theta_a(\cdot)$ of the inverse local time $l^{-1}_\theta(t,0)$ of $X^{(\theta)}$ at zero, can be calculated as $\nu^{(\theta)}(du) = \int e^{-ul}\tau^{(\theta)}_{a^{(\theta)}}(dl) = e^{\theta l}\nu(dy)$ and for the Laplacian of $l^{-1}_\theta(t,0)$ we get

$$E^{(\theta)}_0 \exp[\lambda l^{-1}_\theta(t,0)] = \exp\left(-\frac{t}{G_\theta(\lambda)}\right), \quad \lambda<0, t>0.$$

3.7. Remark: If we do not fix $c(\theta)=[h^{(\theta)}(0)]^{-1}$, we obtain

$$\begin{aligned}
\frac{1}{G^{(\theta)}(\lambda)} &= c(\theta)h_\theta(0)\left[\frac{1}{G(\lambda+\theta)}-\frac{1}{G(\theta)}\right],\\
\frac{1}{a_\theta} &= c(\theta)h_\theta(0)\left(\frac{1}{a}+\frac{1}{G(\theta)}\right),\\
\nu^{(\theta)}(du) &= e^{\theta u}\nu(du)\cdot c(\theta)h^{(\theta)}(0).
\end{aligned}$$

This shows that the quantities a_θ, $G_\theta(\cdot)$ and $\nu^{(\theta)}(\cdot)$ as well as $m^{(\theta)}$ and $s^{(\theta)}$ are defined up to a constant $c^{(\theta)}$ for every process $X^{(\theta)}$, but this constant can vary with θ. What we have shown is that there exists an appropriate choice of $c(\theta)$ such that we get a simple connection between these quantities for different θ.
What is independent of the choice of $c(\theta)$ is e.g. for every $\varepsilon>0$

$$\frac{\nu^{(\theta)}(du)}{\nu^{(\theta)}((\varepsilon,\infty))} = \frac{e^{\theta u}\nu(du)}{\int_\varepsilon^\infty e^{\theta v}\nu(dv)} =: Q^{(\theta)}_\varepsilon(du), \quad u>\varepsilon,$$

which express the conditional probability of the length of an excursion of $X^{(\theta)}$ from 0 given that this excursion is longer than ε. Now the following Corollary is obvious.

3.8. Corollary: *For every choice of* $c(\theta)$ *and for every fixed* ε *with* $\nu(\varepsilon,\infty)>0$ *the probabilities* $\{Q^{(\theta)}_\varepsilon, \theta\le 0\}$ *form an exponential family of distributions.*

4 Exponential families related to $X^{(\theta)}$

Let $X=(\Omega,\mathcal{F},X_t,\mathcal{F}_t,\zeta,P_x)$ be a diffusion on $[0,L)$ as above. Define

$$h^{(\theta)}(x) := E_x e^{\theta\zeta}, \quad x\in[0,L),\ \theta\le 0$$

and consider the family $(X^{(\theta)},\theta\le 0)$ of diffusions corresponding to the transition functions

$$P^{(\theta)}(t,x,dy) = e^{\theta t}P(t,x,dy)\frac{h^{(\theta)}(y)}{h^{(\theta)}(x)}.$$

For abbreviations of the notations we shall put $\tau_a(A)=0$ for every Borel-set $A\subset(-\infty,0)$.

4.1. Theorem: *For the family* $(X^{(\theta)},\theta\le 0)$ *of diffusions the following properties hold:*

(i) *For every $t > 0$, $(t \wedge \zeta, X_{t\wedge\zeta})$ is a sufficient statistic for θ with respect to $\mathcal{F}_t$, for every P_x.*

(ii) *Assume $0 \leq y < x < L$. Then the first hitting time distributions*

$$\left(P_x^{(\theta)}(\sigma_y \in dt)\right), \quad \theta \leq 0$$

form an exponential family of distributions:

$$P_x^{(\theta)}(\sigma_y \in dt) = \frac{\exp(\theta t) P_x(\sigma_y \in dt)}{\int\limits_0^\infty \exp(\theta s) P_x(\sigma_y \in ds)}. \tag{4.1}$$

(iii) *For every $x \in [0, L)$ the life-time distributions*

$$(P_x^{(\theta)}(\zeta \in dt), \theta \leq 0)$$

form an exponential family of distributions:

$$P_x^{(\theta)}(\zeta \in dt) = \frac{\exp(\theta t) P_x(\zeta \in dt)}{h^{(\theta)}(x)}.$$

If $x = 0$ then we have from (2.10) and (2.18)

$$P_0^{(\theta)}(\zeta \in dt) = \frac{1}{a_\theta} \int\limits_0^\infty \mu e^{-\mu t} \frac{\tau_a(d\mu + \{\theta\})}{\mu}\, dt, t > 0, \quad \theta \leq 0, \tag{4.2}$$

i.e., ζ has a mixed exponential distribution under $P_0^{(\theta)}$ too, and the mixing (probability) measure is

$$\frac{1}{a_\theta} \cdot \frac{\tau_a(d\mu + \{\theta\})}{\mu}, \quad \mu \geq 0,$$

with $\frac{1}{a_\theta} = \frac{1}{a} + \frac{1}{G(\theta)}$.

(iv) *The inverse local times $(l_\theta^{-1}(t, 0), t \geq 0)$ of $X^{(\theta)}$ at zero form an exponential family of processes with independent stationary increments, and their Lévy-measures are ν_θ given by*

$$\nu_\theta(du) = e^{\theta u} \nu(du) = e^{\theta u} \int\limits_0^\infty e^{-us} \tau_0(ds). \tag{4.3}$$

$(l_\theta^{-1}(t, 0))$, $t \geq 0$ is killed with constant killing rate a_θ.

(v) *The spectral measure τ_{a_θ} of (m_θ, s_θ) is a shift of τ_a:*

$$\tau_{a_\theta}^{(\theta)}(u) = \tau_a(u + \theta), \quad u \geq 0. \tag{4.4}$$

Proof: (i): Because of the definition of $h^{(\theta)}(\cdot)$ we have

$$\frac{dP_x^{(\theta)}}{dP_x}(\omega) = \frac{\exp(\theta \zeta(\omega))}{h^{(\theta)}(x)}, \quad x \in [0, L), \tag{4.5}$$

and for the restrictions $P_{x,t}^{(\theta)}$ and $P_{x,t}$ of $P_x^{(\theta)}$ and P_x, respectively, to $\mathcal{F}_t$ it holds

$$\frac{dP_{x,t}^{(\theta)}}{dP_{x,t}}(\omega) = \frac{\exp[\theta(\zeta \wedge t)] \cdot h^{(\theta)}(X_{t\wedge\zeta})}{h^{(\theta)}(x)} \tag{4.6}$$

where in this formula is assumed that

$$h^{(\theta)}(X_\zeta) = 1 \quad P_x - \text{a.s. for all } x \in [0, L).$$

For the proof the reader is refered to Asmussen (1989).
(ii): If $y < x$ then by (2.3) it follows

$$E_x^{(\theta)} \exp[\lambda\sigma_y] = \frac{\chi^{(\theta)}(y,\lambda)}{\chi^{(\theta)}(x,\lambda)}, \quad \lambda < 0, \theta \leq 0. \tag{4.7}$$

From (2.2a), (3.7) and (3.8) it follows, that this expression equals

$$\frac{\chi(x,\lambda+\theta)}{\chi(y,\lambda+\theta)} \Big/ \frac{\chi(x,\theta)}{\chi(y,\theta)}.$$

Applying (2.3) once again we obtain that

$$E_x^{(\theta)} \exp[\lambda\sigma_y] = \frac{\int\limits_0^\infty e^{\lambda t} e^{\theta t} P_x(\sigma_y \in dt)}{\int\limits_0^\infty e^{\theta t} P_x(\sigma_y \in dt)}.$$

Now (ii) is obvious.
(iii): From (4.5) it follows

$$E_x^{(\theta)} e^{\lambda\zeta} = \frac{E_x\left(e^{(\lambda+\theta)\zeta}\right)}{h^{(\theta)}(x)} = \frac{h^{(\theta+\lambda)}(x)}{h^{(\theta)}(x)}, \; x \in [0,L), \lambda < 0, \theta \leq 0.$$

Thus, $(P_x^{(\theta)}(\zeta \in dt), \theta \leq 0)$ forms an exponential family:

$$P_x^{(\theta)}(\zeta \in dt) = \frac{e^{\theta t} P_x(\zeta \in dt)}{h^{(\theta)}(x)}, \qquad x \in [0,L),\; t > 0. \tag{4.8}$$

For $x = 0$ we get from (2.18)

$$P_0^{(\theta)}(\zeta \in dt) = a \cdot \exp(\theta t)\left(\frac{1}{a} + \frac{1}{G(\theta)}\right) P_0(\zeta \in dt) = \exp(\theta t - \ln\frac{a_\theta}{a}) \cdot P_0(\zeta \in dt).$$

Using Corollary 2.11 (i) and $\frac{1}{a_\theta} = \frac{1}{a} + \frac{1}{G(\theta)}$ we get (4.2). Because of (4.8) and the fact, that the life-time distribution under $P_x^{(\theta)}$ is the convolution of the hitting time distribution of σ_0 under $P_x^{(\theta)}$ and the life-times distribution under $P_0^{(\theta)}$ (see Corollary 2.11 (ii)), we obtain now that

$$(P_x^{(\theta)}(\zeta \in dt),\; \theta \leq 0)$$

forms an exponential family of distributions also.
(iv) It suffices to show that for every $t > 0$ the distributions of $l^{-1}(t,0)$ under $P_0^{(\theta)}$ form an exponential family. To do this consider

$$E_0^{(\theta)} e^{\lambda l^{-1}(t,0)} = e^{-\frac{t}{G_\theta(\lambda)}} = \exp\left[-t\left(\frac{1}{G(\lambda+\theta)} - \frac{1}{G(\theta)}\right)\right] = \frac{e^{-\frac{t}{G(\lambda+\theta)}}}{e^{-\frac{t}{G(\theta)}}}.$$

(v) Was already proved in Chapter 3. $\square$

4.2. Remark: It is possible to derive a spectral representation for the distribution of σ_0 under P_x in terms of the spectral measure τ_0, see Küchler, Salminen (1989).

The result of Theorem 4.1 (iv) does not remain valid if X can be killed at more than one place. To illustrate it, let us assume additionally that $s(L-) + m(L-) < \infty$, i.e. L is a regular boundary in Fellers terminology. Moreover, let h be the constant occuring in the boundary condition at L:

$$h \cdot D_s^+ f(L) + f(L) = 0.$$

Then we have

4.3. Proposition:

$$h^{(\theta)}(x) = E_x \exp[\theta\zeta] = \frac{1}{h} r_\theta(x, L) + \frac{1}{a} r_\theta(x, 0) =$$
$$\frac{\chi(x,\theta)}{\chi(0,\theta)} \cdot \frac{\frac{1}{a}}{\frac{1}{a} + \frac{1}{G(\theta)}} + \frac{\varphi_a(x,\theta)}{\varphi_a(L,\theta)} \frac{\frac{1}{h}}{\frac{1}{h} + \frac{1}{H(\theta)}}, \ \theta \le 0, x \in (0, L).$$

Here G and H denote the characteristic function of (m, s) together with the boundary constant h and of $(m(L - \cdot) - m(L-), s(L - \cdot) - s(L))$ together with the boundary constant a, respectively.

The proof is very similar to those of Lemma 2.10 and omitted.
Now $h^{(\theta)}$ is of more complicate form and one can not expect that $(l_\theta^{-1}(t, 0),$
$t > 0)$ form an exponential family as above. But from the general theory (Asmussen (1989)) it follows, that the distributions of the life-time ζ under P_x^θ still form an exponential family.

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